Chapter 3: Complex Probability Analysis

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Abstract

This chapter introduces complex probability into digital circuit analysis. Although the eventual use of modification of probability to make room for complex numbers, is still unknown, the mathematics and properties of such numbers, is beautiful.

1 Complex Probability

You may wonder, the physical interpretation of complex probability. Given that it is complex probability, if it will relate to real world analysis. Answer to this question, remains, in understanding the definition of probability.

Conventional probability is defined as

$$\text{probability} = \frac{\text{instances true}}{\text{number of trials}} \tag{1}$$

In our formulation, we will remain true to this definition, expect for the matter that, we will analyze events, in the fourier domain. As you probably know,a normalized $|F(\omega_x)|$ of frequency ω_x, is the probability of finding frequency ω_x,in a signal and $\angle F(\omega_x)$ is its phase, of that frequency, in the signal.

Given circuit function $\mathrm{C}(x_1, x_2, x_3, ..., x_n)$ where x_n are binary independent variables values. Now,if we take the expected value of $\mathrm{C}(\mathbf{X})$, then

$$\mathrm{E}\left[\mathrm{C}(\mathbf{X})\right] \quad = \quad \left[\prod_i^n \int_{-\infty}^{\infty} dx_i \mathrm{pdf}(x_i)\right] \mathrm{C}(\mathbf{X}) \tag{2}$$

The $\mathrm{pdf}(x_i)$ for the independent binary variable x_i is nothing but,

$$\mathrm{pdf}(x_i) = (1 - p_i)\delta(x_i) + p_i\delta(1 - x_i) \tag{3}$$

where p_i is the probability of x_i being a 1. And $\delta(x)$ is the dirac delta[1]. Dirac deltas also have this property that

$$\int_{-\infty}^{\infty} dx\delta(x - a)f(x) = f(a) \tag{4}$$

[1]Dirac deltas are impulse functions that extend to infinity.

We can imagine the $\text{pdf}(x_n)$ concentrated at 0 and 1 with amplitudes $1 - p_n$ and p_n respectively. This formulation is consistant with the Parker-McClusky formulation. For example, the dropping of the exponent can be derived here.

$$\begin{aligned}
\mathrm{E}[x^n] &= \int_{-\infty}^{\infty} dx \left[(1-p_x)\delta(x) + p_x\delta(1-x)\right] x^n && (5)\\
&= \int_{-\infty}^{\infty} dx(1-p_x)\delta(x)x^n + \int_{-\infty}^{\infty} dx p_x\delta(1-x)x^n && (6)\\
&= 0 + p_x && (7)\\
& && (8)
\end{aligned}$$

2 The Newer Definition of $\mathrm{E}\left[\mathrm{C}(\mathbf{X})\right]$

Now, we define

$$\Omega\left[\mathrm{C}(\mathbf{X})\right] = \left[\prod_i^n \int_{-\infty}^{\infty} dx_i \mathrm{pdf}(x_i) e^{-\imath\omega_i x_i}\right] \mathrm{C}(\mathbf{X}) \quad (9)$$

is the spatial frequency distribution of the logic function $\mathrm{C}(\mathbf{X})$. You may ask, aren't fourier transforms taken over time. Yes. but, this fourier transform is not taken over time, but taken over space. Basically, it represents the average of frequency responses emanating from the spatial distribution of impluses.The function defined here this way, is directly related to the characteristic function $\phi[\mathrm{C}(\mathbf{X})]$, we will return to it later.

Now, let's iterate through how the Parker-Mulkusly expressions change under this definition.

$$\begin{aligned}
\Omega[1] &= \int_{-\infty}^{\infty} dx \left[(1-p_x)\delta(x) + p_x\delta(1-x)\right] e^{-\imath\omega_x x} 1 && (10)\\
&= (1-p_x) + p_x e^{-\imath\omega_x} && (11)
\end{aligned}$$

$$\begin{aligned}
\Omega[0] &= \int_{-\infty}^{\infty} dx \left[(1-p_x)\delta(x) + p_x\delta(1-x)\right] e^{-\imath\omega_x x} 0 && (12)\\
&= 0 && (13)
\end{aligned}$$

$$\begin{aligned}
\Omega[x^n] &= \int_{-\infty}^{\infty} dx[(1-p_x)\delta(x) + p_x\delta(1-x)]e^{-\imath w_x x} x^n && (14)\\
&= p_x e^{-\imath\omega_x} && (15)
\end{aligned}$$

From the above, we can gather that,

$$\Omega[1 - x^n] = 1 - p_x \quad (16)$$

You should notice that these expression become Parker-Mclucksy ones when we set ω_x to 0. The expressions here, are bit a unfriendly and non-syememtric for 0 and 1 case. So, So, we redefine our $\Omega\left[\mathrm{C}(\mathbf{X})\right]$ definition.

$$\Omega\left[\mathrm{C}(\mathbf{X})\right] = \left[\prod_i^n \int_{-\infty}^{\infty} dx_i \mathrm{pdf}(x_i) e^{-\imath\omega_i x_i - \imath\theta_i}\right] \mathrm{C}(\mathbf{X}) \quad (17)$$

Now, with this new definition, the expressions become,

$$\Omega[1] = \int_{-\infty}^{\infty} dx\left[(1-p_x)\delta(x) + p_x\delta(1-x)\right] e^{-\imath\omega_x x - \imath\theta_x} 1 \quad (18)$$

$$= (1-p_x)e^{-\imath\theta_x} + p_x e^{-\imath\omega_x - \imath\theta_x} \quad (19)$$

$$\Omega[0] = \int_{-\infty}^{\infty} dx\left[(1-p_x)\delta(x) + p_x\delta(1-x)\right] e^{-\imath\omega_x x - \imath\theta_x} 0 \quad (20)$$

$$= 0 \quad (21)$$

$$\Omega[x^n] = \int_{-\infty}^{\infty} dx[(1-p_x)\delta(x) + p_x\delta(1-x)]e^{-\imath w_x x - \imath\theta_x} x^n \quad (22)$$

$$= p_x e^{-\imath\omega_x - \imath\theta_x} \quad (23)$$

Now,

$$\Omega[1-x^n] = (1-p_x)e^{-\imath\theta_x} \quad (24)$$

It should noted that $|\Omega[\mathrm{C}(\mathbf{X}]|$ remains the same with the old and new definition. The proof is trival, because the absolute value of the multiplicative complex factor $\prod_i^n e^{-\imath\theta_i}$ is 1.

Now, suppose we replace $(1-p_x)e^{-\imath\theta_x}$ with X_0 and $p_x e^{-\imath\omega_x - \imath\theta_x}$ with X_1, our expressions,get very interesting. Notice that

$$|X_0| + |X_1| = 1 \quad (25)$$

$$\Omega[0] = 0 \quad (26)$$

$$\Omega[1-x^n] = X_0 \quad (27)$$

$$\Omega[x^n] = X_1 \quad (28)$$

$$(29)$$

X_1 is just a style of notation, not necessarily, X at index 1.

To construct a complex probability function,from a boolean function, one can do the following, given that boolean function is written as product of sums. Replace instances of x'_n with x_n^0 and x_n with x_n^1. For example, $\vee(x,y) = xy + x'y + xy'$. Now, the complex $\vee(x,y)$ function isx

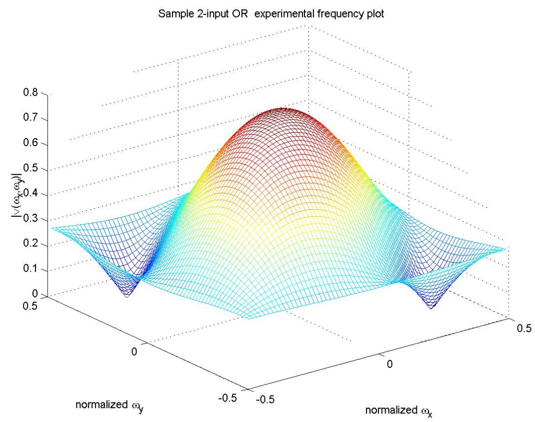
$$\vee(X,Y) = X_1Y_1 + X_0Y_1 + X_1Y_0 \quad (30)$$

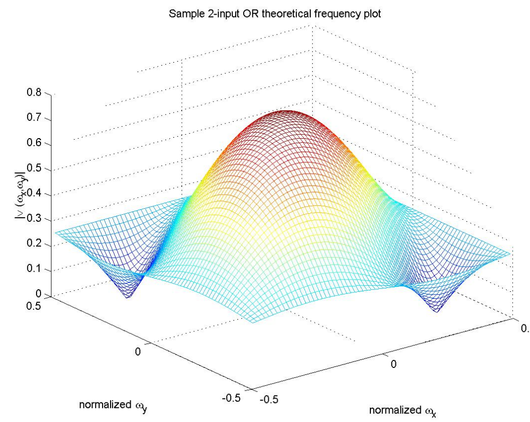
Sample Plots

The following plots were done for a 2-input function with inputs x and y where p_x and p_y are set to 0.5. The $g(p_x e^{-\imath\omega_x}, p_y e^{-\imath\omega_y})$ function is defined as

$$g(x,y) = (1-p_x)(1-p_y) + (1-p_x)p_y e^{-\imath\omega_y} + p_x(1-p_y)e^{-\imath\omega_x}$$

where $-\pi \leq \omega_x \leq \pi$ and $-\pi \leq \omega_y \leq \pi$





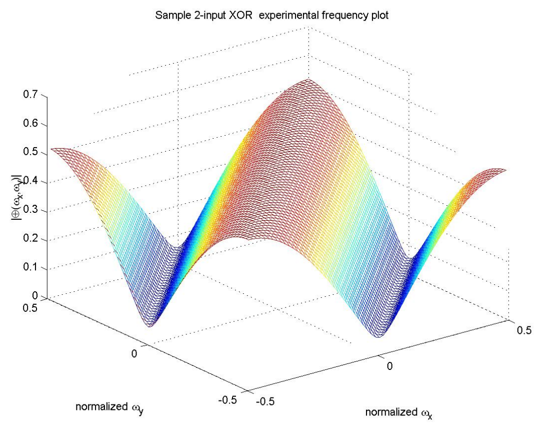
Conducting Experiments

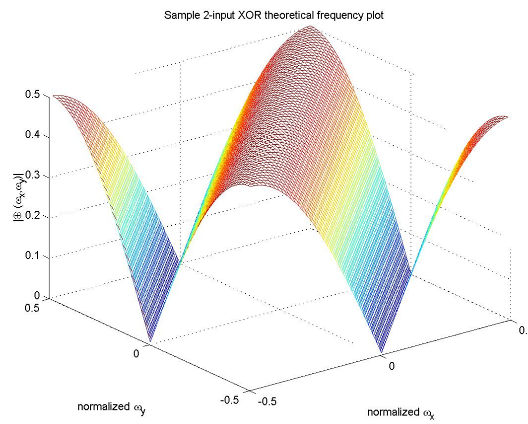
Here's a sample piece of code that verifies the spatial spectral of an XOR gate.

```
% Intialize
% set p_x = 0.5 and p_y = 0.5
Xp = .5;
Yp = .5;
% These variables will be explained later
resolution = 64;
sum = zeros(resolution,resolution);
n = 1000;
% Start Experiment for n iterations
for i =1:n
X = rand(1,1) < Xp;
Y = rand(1,1) < Yp;
% Evaluate function
F =  (1-X) .* (Y) + (1-Y) .* X;

% We contruct a matrix
% x\y  0   1
% 0    1   1
% 1    1   1
```

```
% and  mask that matrix with F
%
% So, when the function turns into a one, the
% repective location in to matrix turns into a
% 1

M = [  (1-X) * (1-Y)*F  (1-X) * Y * F ;
  X*(1-Y)*F  X*Y*F];

% We extend the matrix by appending a lot of zeros
Membedded = [ M  zeros(2,resolution-2);
zeros(resolution-2, resolution)];

%  We take the fft and sum the result
sum = sum + fftshift(fft2(Membedded));
end;
% Average the result
avg = sum / n;
[Xaxes,Yaxes] = meshgrid(  (1:resolution) /
  resolution -.5, (1:resolution) /
 resolution -.5);
figure(1);
% Graph experimental Result
mesh(Xaxes, Yaxes, abs(avg));
title('Sample 2-input XOR  experimental
frequency plot');
xlabel('normalized \omega_x');
ylabel('normalized \omega_y');
zlabel(' |\oplus(\omega_x,\omega_y)|');

figure(2);
% Graph theorectical Result
Mtheory =   (Yp)*(1-Xp) * exp( -j*2*pi*Xaxes)
+ (Xp)*(1-Yp)* exp(-j*2*pi*Yaxes);

mesh(Xaxes, Yaxes,abs(Mtheory));
title('Sample 2-input XOR theoretical
frequency plot');
xlabel('normalized \omega_x');
ylabel('normalized \omega_y');
zlabel(' |\oplus (\omega_x,\omega_y)|');
```

The output result of this program are





Usage

Currently, we cannot think of any usage other than for Formal Hardware Verification. Formal Hardware Verification algorithms can exploit the spectral distribution of circuits and prove their differences or equivalence.