Chapter 4:Variable RTG

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Abstract

This chapter introduces the idea of embedding orthogonal transforms into binary signals. Some of the interesting results of this chapter is that binary waveforms can even carry sinusoids, wavelets, and all other interesting things that were previously, thought to be only applicable to continuous numbers. The goal of this chapter is to lay foundations for our extensions of the work by G.Giani, S.Sheng, V.D.Agarwal, S.Hiaso. The next chapter takes our results futhur and proves specific results.

1 Impossibility of existance of a real weighset that simulatenously sets AND/OR gate output to 0.5

Proof. Take AND gate for $i > 1$,

$$\prod_{i=1}^{n} X_i = 0.5$$

And take an OR gate,

$$1 - \prod_{i=1}^{n}(1 - X_i) = 0.5$$

Now, rewrite it,

$$\prod_{i=1}^{n}(1 - X_i) = 0.5$$

Now, take the AND expression and rewrite it as,

$$\prod_{i=1}^{n-1} X_i = 0.5/X_n$$

and take the OR expression and rewrite it as,

$$\prod_{i=1}^{n-1}(1-X_i) = 0.5/(1-X_n)$$

Add both equations,

$$\prod_{i=1}^{n-1} X_i + \prod_{i=1}^{n-1}(1-X_i) = 0.5/X_n + 0.5/(1-X_n)$$

set $b = \prod_{i=1}^{n-1} X_i + \prod_{i=1}^{n-1}(1-X_i)$,
now, the expression becomes,

$$b = 0.5\frac{1-Xn+X_n}{(1-X_n)(X_n)}$$

when rewritten, it becomes

$$X_n - X_n^2 = 0.5/b$$
$$X_n^2 - X_n + 0.5/b = 0$$

If the determinant $b^2 - 4ac$ of the above equation,

$$1 - 0.5 \cdot 4/b < 0$$

the solution is complex,and so, no real solution exists.

because $b = \prod_{i=1}^{n-1} X_i + \prod_{i=1}^{n-1}(1-X_i)$ can never be greater than 2, if X_i are weightsets. It makes $2 > b$ which implies $2/b > 1$, $0 > 1 - 2/b$, which proves that no-real solution exists. The case, when

$$\begin{aligned} 1 - 2/b &= 0 && (1) \\ 1 &= 2/b && (2) \\ b &= 2 && (3) \\ & && (4) \end{aligned}$$

This case happen only if $\prod_{i=1}^{n-1} X_i = 1$ and $\prod_{i=1}^{n-1}(1-X_i) = 1$. This specific condition is impossible to be solved using a real weight set. And therefore ends our proof, why the one cannot set both an n-input AND gate and an n-input OR gate, to 0.5. $\square$

2 An 'Unconventional' solution

Now imagine a 10 1-bit square waves with the same frequency. Now, since all of them are 1 simulatenously and are 0 too at the same time, they will get the output probability of both the 10-input AND gate and 10-OR gate to 0.5.
It should be noted that this solution does not translate itself to a better weight set since the inputs are allowed only to oscillate between 1111...1 and 0000...0. Anyways, it does go on to show that extending a static weight set to a variable one does allow us to solve problems that are not easily realizable using ordinary static weight sets.

3 Variable RTG:A Spectral Set

We try to explore, interesting consequences, of extending the idea presented in the above section.
Consider the following example with a logic AND gate Z with inputs X and Y. Using Parker Mulclusky equations, we can represent Z as,

$$Z = X \cdot Y \tag{5}$$

To demonstrate this idea, we assign

$$X_{\text{prob}}(t) = A_X + B_X \sin(2\pi f_X t) \tag{6}$$

$$Y_{\text{prob}}(t) = A_Y + B_Y \sin(2\pi f_Y t) \tag{7}$$

As we already know, X is a real, restricted to the interval [0,1] and so is Y. Since, these are probabilistic real variables, to generate their binary counter parts, we just take,

$$X_{\text{binary}}(t) = \text{rand}() < X_{\text{prob}}(t) \tag{8}$$

$$Y_{\text{binary}}(t) = \text{rand}() < Y_{\text{prob}}(t) \tag{9}$$

Now, you may well ask, are the induced frequency effects observable on $X_{\text{binary}}(t)$ and $Y_{\text{binary}}(t)$. Yes, they are, if we take the fft of the signal over time.

$$X_{\text{fft}}(f) = \text{fft}(X_{\text{binary}}(t)) \tag{10}$$

$$Y_{\text{fft}}(f) = \text{fft}(Y_{\text{binary}}(t)) \tag{11}$$

Here are sample normalized plots of variables

$$X = \text{rand}() < .5 + .3\sin(.4\pi t) \tag{12}$$

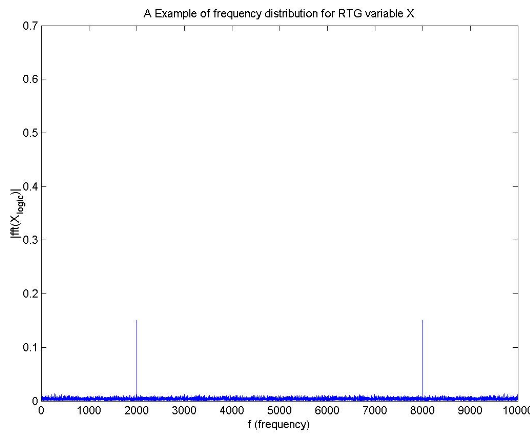
$$Y = \text{rand}() < .5 + .3\sin(.6\pi t) \tag{13}$$

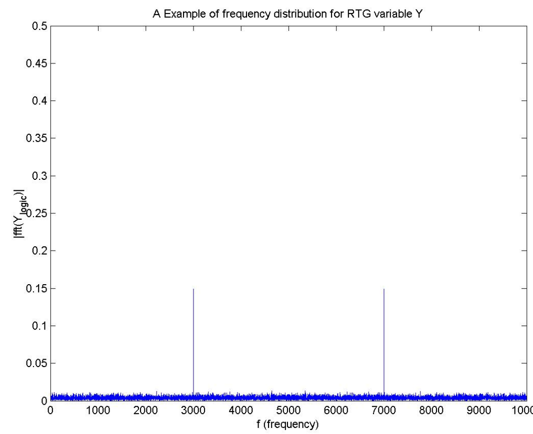
and the resultant variable

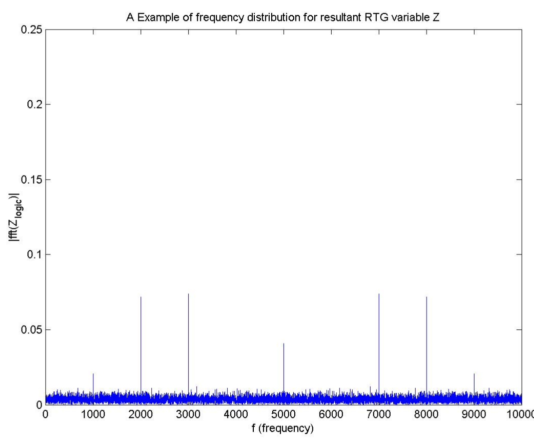
$$Z = X \cdot Y \tag{14}$$

for 10,000 samples.

3.1 Results







We observe frequency peaks surrounded by noise. We also observe constructive and destructive interference of the input frequencies. The results should be of no surprise, to people, who are familar with spectral testing. The apparent noise barges in, during the turnication of the 'real' input variable to a binary random input variable. The results are pretty encouraging because the frequency effects are visible. And indeed,as you might guess, the destructive interference from the frequency components can sometimes affect even the overall probability of output. Later on,we will try to use these ideas to evolve a spectral testset.

4 Proof by Parker-Mcklusky equations and Monto Carlo

Now, the pdf of a binary variable x_i is given by

$$\mathrm{pdf}(x_i) = (1 - p_i)\delta(x_i) + p_i\delta(1 - x_i) \tag{15}$$

where $\delta(x_i)$ is the Dirac delta. Now, suppose, probability of x_i, being a 1, varies over time. The pdf takes the following form.

$$\mathrm{pdf}(x_i, t) = [1 - p_i(t)]\, \delta(x_i) + p_i(t)\delta(1 - x_i) \tag{16}$$

Now, to calculate the expected value of a combinational binary function C($\mathbf{X}$), at time instant t,

$$\mathrm{E[C}|t](\mathbf{X}) = \left[\prod_i^n \int_{-\infty}^{\infty} dx_i \mathrm{pdf}(x_i, t)\right] \mathrm{C}[\mathbf{X}] \tag{17}$$

Now, suppose the [C]($\mathbf{X}$), is just x^n or x AND itself n times. Then, the expected value, is

$$\begin{aligned} \mathrm{E[C}|t](\mathbf{X}) &= \int_{-\infty}^{\infty} dx\, [1 - p_x(t)]\, \delta(x) + p_x(t)\delta(1 - x)\, [x^n] && (18) \\ &= p_x(t) && (19) \end{aligned}$$

Now,for an AND function of binary variables x and y, we can use the same formulation to prove that it is $p_x(t)p_y(t)$.
There is a Monto Carlo way to make sense of the idea too. Traditionally, we have a weightset $\mathbf{X}$. Now, suppose, $\mathbf{X}$ changes in time, let us call it $\mathbf{X}(t)$. Now, suppose, we stop at some instant t and create many many binary input vector, repeatedly, by taking

$$X_{binary}(t) = \mathrm{rand}() < X(t) \tag{20}$$

Now, the expected value of the circuit's output, can be computed by taking averaging the zeros and the ones at the output. This value, would be the same as the one computed for Parker-Mcklusky expressions for $\mathbf{X}(t)$, at time instant t.

Over an interval of time, the overall output probability, will change.

$$\mathrm{E[C]}(\mathbf{X}) = \frac{1}{t_n - t_0} \int_{t_0}^{t_n} dt \left[\prod_i^n \int_{-\infty}^{\infty} dx_i \mathrm{pdf}(x_i, t)\right] \mathrm{C}[\mathbf{X}] \tag{21}$$

That would be, for a binary variable x with a changing pdf, it becomes $\langle p_x(t)\rangle$. The concept of most interest, is not even these. We shall discuss the idea of orthogonal transformations, in the next section.

5 Orthogonal Transformations

Now, generally speaking, we across orthogonal transformation, a lot in analog circuit theory. The most prelavent transformation is the fourier-transform. Then, there is the laplace transform, the wavelet transform, etc, all with different uses. Now, here are some basic properties of othogonal transforms.
Suppose, there exists $\phi_i(t)$ such that

$$\frac{1}{t_n - t_0} \int_{t_0}^{t_n} dt \phi_i(t) = \delta_{i0} \tag{22}$$

and

$$\frac{1}{t_n - t_0} \int_{t_0}^{t_n} dt \phi_i(t) \phi_j^*(t) = \delta_{ij} \tag{23}$$

where δ_{ij} is 1,only when i and j are equal. Now, interestingly, it can be proven that any square integral function f(t), integrated over a sufficiently large t, can be written as

$$\mathrm{f}(t) = \sum_i a_i \phi_i(t) \tag{24}$$

where a_i is given by

$$\frac{1}{t_n - t_0} \int_{t_0}^{t_n} dt \phi_i^*(t) \mathrm{f}(t) \tag{25}$$

which can also be neatly written as $\langle f(t)\phi_i(t)\rangle$.
Now, we can define

$$\mathrm{E}[\mathrm{C}|\phi_i](\mathbf{X}) = \frac{1}{t_n - t_0} \int_{t_0}^{t_n} dt \phi_i^*(t) \left[\prod_i^n \int_{-\infty}^{\infty} dx_i \mathrm{pdf}(x_i, t) \right] \mathrm{C}[\mathbf{X}] \tag{26}$$

Suppose, we try to tranform the output of an input variable x, ANDed n times, we would get $\langle p_x(t)\phi_i(t)\rangle$

6 Limitations of Transformations

The biggest limitation is enforced by shannon's information theory. That would mean, that if you are using binary input vectors and calculating probabilities, by sampling, the more the sample, the closer, your transformation of ad-hoc data, matches the theoretical one. Covergence, would be an issue, depending on the type of othorgonal transform, the region it is valid in, etc.