Newton's Law of Gravitation : Derivation

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Abstract

In this paper, we try to derive Newton's Law of gravitation as a consequence of Gauss's law and Hamitonian mechanics.

Let's start by assuming that there is a particle of mass m, in the presence of mass M. Assume that particle m has initial velocity v_0
Take the Hamiltionian of particle mass m

$$H_0 = \frac{1}{2}mv_0^2 \tag{1}$$

$$\tag{2}$$

Now, suppose we introduce a petrubative potential V(x).

$$H = \frac{1}{2}mv^2 + V(x) \tag{3}$$

We shall assume that the petrubative potential does not change the total energy of the system. Since energy is conserved,

$$\frac{1}{2}mv^2 + V(x) = H_0 \tag{4}$$

It must be noticed that the following derivation does not directly depend on the above steps. In fact, one can get away, without actually going through the above steps. Instead, the purpose of the above steps, is to justify the introduction of a potential $V(x)$, where $V(x)$ would not violate law of conservation of energy.

Asssume that the particle m follows the path of least action, given by

$$m\frac{dv_i}{dt} = -\frac{d}{dx_i}V(x) \tag{5}$$

This statement makes the claim that the particle of mass m experiences acceleration of $\frac{dv_i}{dt}$ from the potential $V(x)$
Suppose the particle is moving along a surface, where acceleration $a_i = \frac{dv_i}{dt}$ is constant. A simplest form of that surface for a 3d-euclidian space is a sphere.

For simplicity, we will work in spherical coordinates, instead of elucidian cordinates.
Assume that a_r is pointing from particle of mass m towards particle of mass M and dS is pointing away from particle of mass M to particle of mass m, and where r and dr are pointing from particle of mass M to mass m.

$$\int_S -a_r dS = 4\pi GM \tag{6}$$

where S is surface of a sphere. The last statement may strike the reader as being odd. Analogous phenomenon exist in the real world. Take sound for example. Sound could propagate using any medium. It does not need a specific exchange particle to carry it and it decays by the inverse square law.
One way to interpret the above statement is that, particle M causes pertrubational acceleration of magnitude a_r at location of particle m and Gauss'law functions as a mechanism for calculating that effect.

$$\begin{aligned} \int_S -a_r(4\pi r^2) &= 4\pi GM \qquad (7) \\ a_r &= -\frac{GM}{r^2} \qquad (8) \end{aligned}$$

$$-\frac{d}{dr}V(r) = -\frac{GMm}{r^2} \tag{9}$$

Integrating again

$$\int dV(r) = \int dr\frac{GMm}{r^2} \tag{10}$$

We end up with

$$V(r) + C = -\frac{GMm}{r} \tag{11}$$

Requiring V(r) to be 0,at $r = \infty$, we end up with

$$V(r) = -\frac{GMm}{r} \tag{12}$$

1 Special Relativity : Emergent Phenomenon ?

Assume the following. There are 2 particles u and v with velocities (u_x, u_t) and (v_x, v_t) , in frame S.
Suppose we require the following Energy relations

$$\frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}m(u-v)^2 \tag{13}$$

and

$$\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = \frac{1}{2}m(u-v)^2 \tag{14}$$

and

$$\frac{1}{2}mv^2 + \frac{1}{2}mu^2 = \frac{1}{2}m(u+v)^2 \tag{15}$$

to be true, for self-consistency. We shall introduce constants a and b such that

$$u \cdot v = a^2 v_x u_x + b^2 v_t u_t \tag{16}$$

Directly from this statement, it follows that

$$u^2 = a^2 u_x u_x + b^2 u_t u_t \tag{17}$$

and

$$v^2 = a^2 v_x v_x + b^2 v_t v_t \tag{18}$$

The only way the above relationship would only hold, if $LHS = RHS = 0$. And if $\frac{1}{2}mu^2 = \frac{1}{2}mv^2$. For any arbitary u and v the relationships would only be true iff

$$a^2 u_x u_x + b^2 u_t u_t = 0 \tag{19}$$

and

$$a^2 v_x v_x + b^2 v_t v_t = 0 \tag{20}$$

and

$$a^2 v_x u_x + b^2 v_t u_t = 0 \tag{21}$$

Setting $\wedge' = \frac{b}{a}$, we have

$$u_x u_x + \wedge'^2 u_t u_t = 0 \tag{22}$$

and

$$v_x v_x + \wedge'^2 v_t v_t = 0 \tag{23}$$

and

$$v_x u_x + \wedge'^2 v_t u_t = 0 \tag{24}$$

Notice that if $\wedge'^2$ is positive, u^2 will never equal to 0. Instead if we replace $\wedge'^2$ with $-\wedge^2$, the expressions will work.

$$u_x u_x - \wedge^2 u_t u_t = 0 \tag{25}$$

and

$$v_x v_x - \wedge^2 v_t v_t = 0 \tag{26}$$

and

$$v_x u_x - \wedge^2 v_t u_t = 0 \tag{27}$$

Also according to these expressions, the kinetic energy of the particle is 0, since $v^2 = 0$, even though it has velocity along x and t components. Interestingly the particle is both moving and in an interial frame of rest. These equations will hold to be true in any frame S'.
From there, our redefine the $u \cdot v$ as

$$u \cdot v = u_x v_x - \wedge^2 u_t v_t \tag{28}$$

2 Introducting c

Let indroduce a directional velocity vector $\mathbf{c}$ (c_x, c_y) with property $c_x^2 - \wedge^2 c_t^2 = 0$. Now, suppose, we mix $\mathbf{c}$ with p_-, probability of moving in the negative direction, p_+ probability of moving in the positive direction, p_0, probability of "no motion". Let

$$\begin{aligned} u &= p_{u+}\mathbf{c} + p_0 0 + p_{u-}(-\mathbf{c}) & (29) \\ &= [p_{u+} - p_{u-}]\,\mathbf{c} & (30) \end{aligned}$$

And let

$$\begin{aligned} v &= p_{v+}\mathbf{c} + p_0 0 + p_{v-}(-\mathbf{c}) & (31) \\ &= [p_{v+} - p_{v-}]\,\mathbf{c} & (32) \\ & & (33) \end{aligned}$$

It follows that

$$\begin{aligned} u^2 &= [p_{u+} - p_{u-}]^2\,\mathbf{c}^2 & (34) \\ v^2 &= [p_{v+} - p_{v-}]^2\,\mathbf{c}^2 & (35) \\ u \cdot v &= (p_{u+} - p_{u-})(p_{v+} - p_{v-})\mathbf{c}^2 & (36) \end{aligned}$$

From the above expressions,we can see that the amount of motion in time interval τ is

$$\begin{aligned} \sqrt{-\frac{u^2}{-\mathbf{c}^2}}\lambda &= \sqrt{[p_{u+} - p_{u-}]^2}\lambda & (37) \\ \sqrt{-\frac{v^2}{-\mathbf{c}^2}}\lambda &= \sqrt{[p_{v+} - p_{v-}]^2}\lambda & (38) \\ & & (39) \end{aligned}$$

But the above statement is nonsense is $-\mathbf{c}^2 = 0$, but u^2 carries $\mathbf{c}^2$ of it own. Allowing them to cancel out, we can safely work with the above expressions.
$\sqrt{[p_{u+} - p_{u-}]^2}\lambda$ and $\sqrt{[p_{v+} - p_{v-}]^2}\lambda$ will remain frame-invariant, [to be filled].